J. Appl. Maths Mechs Vol. 57, No. 3, pp. 511–523, 1993
Printed in Great Britain.

0021–8928/93 $24.00+.00

THE RADIATION AND DIFFRACTION OF STEADY-STATE ELASTIC WAVES IN A PERIODICALLY PERFORATED ORTHOTROPIC PLANE†

V. M. FOMIN

Odessa

(*Received* 21 *April* 1992)

Methods presented in [1] are used to investigate the radiation and diffraction of elastic waves by a periodic row of circular apertures in an orthotropic plane.

A RANGE of similar two-dimensional problems for an isotropic medium has been considered previously [2–4]. A similar problem has been studied for the anisotropic case, apparently only for pure-shear SH-waves in a layer [5].

1. Consider a homogeneous orthotropic elastic medium and let the x_1, x_2, x_3 axes be perpendicular to the planes of elastic symmetry, i.e. the principal axes. We shall investigate the case of plane deformation in the x_1, x_2 plane.

The equations of the dynamic theory of elasticity can be written as

$$\rho\frac{\partial^2 u_k}{\partial t^2} = c_{kjlm}\frac{\partial^2 u_m}{\partial x_j \partial x_l} \tag{1.1}$$

Here u_k is the displacement of points in the direction of the x_k axes, ρ is the density, and c_{kjlm} are the components of the elasticity tensor [6]. In (1.1) and below summation is assumed over repeated indices taking the values 1 and 2.

Putting $u_k = u_k^0\exp(-i\omega t)$ and omitting the time factor, we can write (1.1) in the form

$$\mathbf{C}_{jl}\frac{\partial^2 \mathbf{u}}{\partial x_j \partial x_l} + \rho w^2\mathbf{u} = 0 \tag{1.2}$$

Here $\mathbf{u}$ is a two-dimensional vector with components u_k^0 $(k = 1, 2)$, and $\mathbf{C}_{jl}$ is the matrix with elements c_{kjlm} $(k, m = 1, 2)$.

Green's matrix $\mathbf{G}(\omega, x_1, x_2)$ of the orthotropic plane is the solution of the equation

$$\mathbf{L}\mathbf{G}(\omega, x_1, x_2) = \delta(x_1)\delta(x_2)\mathbf{I}, \quad \mathbf{L} = \mathbf{C}_{kj}\frac{\partial^2}{\partial x_k \partial x_j} + \rho\omega^2\mathbf{I} \tag{1.3}$$

($\delta(x_j)$ is the delta function and $\mathbf{I}$ is the two-dimensional identity matrix).

After performing a double Fourier transformation with respect to x_1 and x_2 in (1.3) we have

$$\mathbf{M}(\omega, \xi_1, \xi_2)\mathbf{G}(\omega, \xi_1, \xi_2) = \mathbf{I}, \quad \mathbf{M}(\omega, \xi_1, \xi_2) = -\mathbf{C}_{kj}\xi_k\xi_j + \rho\omega^2\mathbf{I} \tag{1.4}$$

†*Prikl. Mat. Mekh.* Vol. 57, No. 3, pp. 112–124, 1993.

Here ξ_j $(j=1, 2)$ are the transformation parameters and $\mathbf{G}(\omega, \xi_1, \xi_2)$ is the double Fourier transform of Green's matrix $\mathbf{G}(\omega, x_1, x_2)$. Below we shall denote functions and their transforms by the same letter, distinguishing them by their arguments.

From (1.4) we obtain $\mathbf{G}(\omega, \xi_1, \xi_2)=\mathbf{M}(\omega, \xi_1, \xi_2)^{-1}$. After simplification, this equality can be reduced to the form

$$\begin{aligned} &\mathbf{G}(\omega,\xi_1,\xi_2)=\frac{1}{\Delta(\omega,\xi_1,\xi_2)}(-\mathbf{F}_{kj}\xi_k\xi_j+\rho\omega^2\mathbf{I}) \\ &\Delta(\omega,\xi_1,\xi_2)=\det\mathbf{M}(\omega,\xi_1,\xi_2) \end{aligned} \tag{1.5}$$

where $\mathbf{F}_{kj}$ is the matrix adjoint to $\mathbf{C}_{kj}$ and $\mathbf{F}_{kj}=\det\mathbf{C}_{kj}\cdot\mathbf{C}_{kj}^{-1}$.

Performing an inverse Fourier transformation with respect to ξ_2 in (1.5), we have

$$\mathbf{G}(\omega,\xi_1,x_2)=-\mathbf{F}_{11}\xi_1^2\eta_0(\omega,\xi_1,x_2)-i(\mathbf{F}_{12}+\mathbf{F}_{21})\xi_1\eta_1(\omega,\xi_1,x_2)+\mathbf{F}_{22}\eta_2(\omega,\xi_1,x_2) \tag{1.6}$$

$$\eta_0(\omega,\xi_1,x_2)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\exp(-i\xi_2x_2)}{\Delta(\omega,\xi_1,\xi_2)}d\xi_2,\quad \eta_j=\frac{\partial^j\eta_0}{\partial x_2^j}\quad (j=1,2) \tag{1.7}$$

It can be shown that the equation

$$\Delta(\omega,\xi_1,\xi_2)=0$$

can be represented in the form

$$a_0\xi_2^4+a_1(\omega,\xi_1)\xi_2^2+a_2(\omega,\xi_1)=0 \tag{1.8}$$

and in the case of an orthotropic plane

$$\begin{aligned} &a_0=c_{22}c_{66},\quad a_1(\omega,\xi_1)=a_{11}\xi_1^2-a_{12}\omega_0^2 \\ &a_2(\omega,\xi_1)=a_{21}\xi_1^4-a_{22}\xi_1^2\omega_0^2+\omega_0^4,\quad \omega_0^2=\rho\omega^2 \\ &a_{11}=c_{11}c_{22}-c_{12}^2-2c_{12}c_{66},\quad a_{12}=c_{22}+c_{66},\quad a_{21}=c_{11}c_{66} \\ &a_{22}=c_{11}+c_{66},\quad c_{11}=c_{1111},\quad c_{22}=c_{2222},\quad c_{12}=c_{1122},\quad c_{66}=c_{1212} \end{aligned}$$

For orthotropic materials, whose elastic constants were given in [6, 7], the bilinear form takes positive values. Then the roots ξ_{2j} $(j=1, 2, 3, 4)$ of Eq. (1.8) lie on the real and imaginary axes, symmetrically with respect to those axes (Fig. 1).

To evaluate integral (1.7) we use Jordan's lemma and the residue theorem. The contour of integration for the case when $x_2>0$ is shown in Fig. 1. We obtain

$$\eta_0(\omega,\xi_1,x_2)=-\frac{1}{2a_0(\gamma_2^2-\gamma_1^2)}\sum_{j=1}^{2}(-1)^{j-1}\frac{\exp(-\gamma_j|x_2|)}{\gamma_j} \tag{1.9}$$

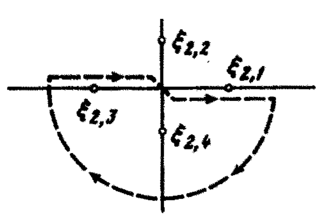

FIG. 1.

$$\gamma_j = \left\{\frac{a_1(\omega,\xi_1)+(-1)^j[a_1(\omega,\xi_1)^2-4a_0a_2(\omega,\xi_1)]^{1/2}}{2a_0}\right\}^{1/2}$$

In accordance with the radiation conditions in the second formula of (1.9) we take the branch of the radical for which $\gamma_j=\sqrt{a}$ for $a\geqslant 0$ and $\gamma_j=-i\sqrt{-a}$ for $a<0$. (Here the square roots are positive.)

Performing an inverse Fourier transformation with respect to ξ_1 in (1.6), we obtain Green's matrix for an orthotropic plane in the form

$$\mathbf{G}(\omega,x_1,x_2)=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-i\xi_1x_1)[\mathbf{F}_{11}\xi_1^2\eta_0(\omega,\xi_1,x_2)+$$
$$+i(\mathbf{F}_{12}+\mathbf{F}_{21})\xi_1\eta_1(\omega,\xi_1,x_2)-\mathbf{F}_{22}\eta_2(\omega,\xi_1,x_2)]d\xi_1 \quad (1.10)$$

2. The stress component parallel to the x_i axis acting across an area element with normal $\mathbf{n}$ is given by the formula $p_i=\sigma_{ik}n_k$ (where the σ_{ik} are the components of the stress tensor and n_k is the projection of $\mathbf{n}$ onto the x_k axis).

Using Hooke's law

$$\sigma_{ik}=\frac{1}{2}c_{iklm}\left(\frac{\partial u_l}{\partial x_m}+\frac{\partial u_m}{\partial x_l}\right)$$

we find after a series of transformations that

$$p_i=\frac{\partial}{\partial x_m}t_{ilm}^{\mathbf{n}}u_l,\quad t_{ilm}^{\mathbf{n}}=c_{iklm}n_k \quad (2.1)$$

Note that relation (2.1) can be written as

$$\mathbf{P}^{\mathbf{n}}=\mathbf{T}^{\mathbf{n}}\mathbf{u},\quad \mathbf{T}^{\mathbf{n}}=\mathbf{T}_m^{\mathbf{n}}\frac{\partial}{\partial x_m},\quad \mathbf{T}_m^{\mathbf{n}}=[t_{ilm}^{\mathbf{n}}]_{i,l=1}^2 \quad (2.2)$$

Acting with the operator $\mathbf{T}^{\mathbf{n}}$ on Green's matrix (1.10) we find that

$$\mathbf{G}_1^{\mathbf{n}}(\omega,x_1,x_2)=\mathbf{T}^{\mathbf{n}}\mathbf{G}(\omega,x_1,x_2)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{G}_1^{\mathbf{n}}(\omega,\xi_1,x_2)\exp(-i\xi_1x_1)d\xi_1$$

The matrix $\mathbf{G}_1^{\mathbf{n}}(\omega,\ x_1,\ x_2)$ will be called Green's stress matrix.

From (2.2) and (1.10) we obtain

$$\mathbf{G}_1^{\mathbf{n}}(\omega,\xi_1,x_2)=\sum_{m=0}^{3}\eta_m(\omega,\xi_1,x_2)\mathbf{U}_m(\omega,\xi_1) \quad (2.3)$$
$$\mathbf{U}_0(\omega,\xi_1)=i\xi_1(\xi_1^2\mathbf{T}_1^{\mathbf{n}}\mathbf{F}_{11}-\rho\omega^2\mathbf{T}_1^{\mathbf{n}})$$
$$\mathbf{U}_1(\omega,\xi_1)=\xi_1^2\mathbf{T}_2^{\mathbf{n}}\mathbf{F}_{11}-\xi_1^2\mathbf{T}_1^{\mathbf{n}}\mathbf{F}_{12}+\rho\omega^2\mathbf{T}_2^{\mathbf{n}}$$
$$\mathbf{U}_2(\omega,\xi_1)=-i\xi_1(\mathbf{T}_2^{\mathbf{n}}\mathbf{F}_{12}+\mathbf{T}_1^{\mathbf{n}}\mathbf{F}_{22})$$
$$\mathbf{U}_3(\omega,\xi_1)=\mathbf{T}_2^{\mathbf{n}}\mathbf{F}_{22}$$

$$\eta_m(\omega,\xi_1,x_2)=\frac{(-1)^m(\operatorname{sign}x_2)^m}{2a_0(\gamma_2^2-\gamma_1^2)}\sum_{j=1}^{2}(-1)^{j-1}\gamma_j^{m-1}\exp(-\gamma_j|x_2|) \quad (2.4)$$

When deriving (2.4) it was assumed that when $m=2$ the factor in front of $\delta(x_2)$ vanishes when $x_2 = 0$ and hence the corresponding term can be omitted.

It can be shown that

$$\gamma_j \sim |\xi_1| d_j, \quad d_j = \{a_{11}^2 + (-1)^j (a_{11}^2 - 4a_0 a_{21})^{1/2}\}^{1/2} \quad \text{as } |\xi_1| \to \infty$$

From this it follows that

$$\eta_m(\omega, \xi_1, x_2) \sim \frac{(-1)^m (\operatorname{sign} x_2)^m}{2a_0(d_2^2 - d_1^2)|\xi_1|^3} \sum_{j=1}^{2} (-1)^j d_j^{m-1} \exp(-d_j |\xi_1 x_2|) \tag{2.5}$$

Using (2.5), we find that $\mathbf{G}_1^n(\omega, \xi_1, x_2) \sim \mathbf{G}_1^n(0, \xi_1, x_2)$ as $|\xi_1| \to \infty$.

After transforming (2.3) we obtain

$$\mathbf{G}_1^n(0, \xi_1, x_2) = \frac{1}{2a_0(d_2^2 - d_1^2)} \sum_{j=1}^{2} (-1)^{j-1} [i \operatorname{sign}(\xi_1) d_j \mathbf{N}_{j1}^n -$$
$$- \operatorname{sign}(x_2) \mathbf{N}_{j2}^n] \exp(-d_j |\xi_1 x_2|) \tag{2.6}$$
$$\mathbf{N}_{j1}^n = d_j^{-2} \mathbf{T}_1^n \mathbf{F}_{11} + \mathbf{T}_2^n \mathbf{F}_{12} - \mathbf{T}_1^n \mathbf{F}_{22}, \quad \mathbf{N}_{j2}^n = \mathbf{T}_1^n \mathbf{F}_{12} - \mathbf{T}_2^n \mathbf{F}_{11} + d_j^2 \mathbf{T}_2^n \mathbf{F}_{22}$$

It can be shown that

$$|\mathbf{G}_1^n(\omega, \xi_1, x_2) - \mathbf{G}_1^n(0, \xi_1, x_2)| = \omega^2 [|x_2| O(|\xi_1|^{-1}) + O(|\xi_1|^{-2})] \tag{2.7}$$

Substituting (2.6) into (2.3) and integrating, we have

$$\mathbf{G}_1^n(0, x_1, x_2) = \frac{1}{2a_0 \pi (d_2^2 - d_1^2)} \sum_{j=1}^{2} (-1)^{j-1} d_j \left(\mathbf{N}_{j1}^n \frac{x_1}{d_j^2 x_2^2 + x_1^2} - \mathbf{N}_{j2}^n \frac{x_2}{d_j^2 x_2^2 + x_1^2} \right) \tag{2.8}$$

We will represent the matrix $\mathbf{G}_1^n(\omega, x_1, x_2)$ in the form

$$\mathbf{G}_1^n(\omega, x_1, x_2) = \mathbf{G}_1^n(0, x_1, x_2) + \mathbf{R}^n(\omega, x_1, x_2)$$
$$\mathbf{R}^n(\omega, x_1, x_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\mathbf{G}_1^n(\omega, \xi_1, x_2) - \mathbf{G}_1^n(0, \xi_1, x_2)] \exp(-i\xi_1 x_1) d\xi_1 \tag{2.9}$$

It follows from estimate (2.7) that the matrix $\mathbf{R}^n(\omega, x_1, x_2)$ is continuous over the combined variables x_1, x_2. Thus the singularities of the functions $\mathbf{G}_1^n(\omega, x_1, x_2)$ when $\omega \neq 0$ and $\mathbf{G}_1^n(0, x_1, x_2)$ coincide. It follows from (2.8) that $\mathbf{G}_1^n(0, x_1, x_2)$ has a unique singularity at $\mathbf{x} = 0$.

We will now construct Green's matrix $\Gamma(\omega, x_1, x_2)$ for the quasi-periodic problem, i.e. we determine the components of the displacement amplitudes at the point $x = (x_1, x_2)$ from the action of the system of lumped forces $\mathbf{e}_{jm} \exp(-i\omega t)$ applied at points with coordinates $x_1 = ml$ $(m = 0, \pm 1, \pm 2, \ldots)$, $x_2 = 0$ (l being the length of some line segment), satisfying the equalities $\mathbf{e}_{jm} = \mathbf{e}_j \exp(-im\alpha)$ ($\mathbf{e}_j$ being the unit vector along the x_j axis, $j = 1, 2$), and from these components we then form the jth column of the matrix $\Gamma(\omega, x_1, x_2)$. Here α is the parameter of the quasi-periodic problem, $|\alpha| \leq \pi$.

Green's stress matrix of the quasi-periodic problem is determined similarly, its columns being components of the stresses over the area element with normal $\mathbf{n}$. It is denoted by $\Gamma_1^n(\omega, x_1, x_2)$.

Representations of dynamic Green's matrices for anisotropic planes have been obtained in a series of papers (see, for example, [8, 9]). However, these representations cannot be used to construct Green's matrix efficiently for a quasi-periodic problem.

We replace x_1 by $x_1 - ml$ in (2.9), multiply it by $\exp(-im\alpha)$ and sum over m from $-\infty$ to ∞.

We obtain

$$\Gamma_1^n(\omega,x_1,x_2)=\Gamma_1^n(0,x_1,x_2)+Q^n(\omega,x_1,x_2) \tag{2.10}$$

$$\Gamma_1^n(0,x_1,x_2)=G_1^n(0,x_1,x_2)+S^n(x_1,x_2)$$

$$S^n(x_1,x_2)=[2a_0\pi(d_2^2-d_1^2)]^{-1}\sum_{j=1}^{2}(-1)^{j-1}d_j\left[N_{j1}^n{\sum_m}'\frac{(x_1-ml)\exp(-im\alpha)}{d_j^2x_2^2+(x_1-ml)^2}-N_{j2}^n{\sum_m}'\frac{x_2\exp(-im\alpha)}{d_j^2x_2^2+(x_1-ml)^2}\right]$$

$$Q^n(\omega,x_1,x_2)=\sum_m R^n(\omega,x_1-ml,x_2)\exp(-im\alpha)$$

$$Q^n(\omega,x_1,x_2)=Q_1^n(\omega,x_1,x_2)-Q_1^n(0,x_1,x_2) \tag{2.11}$$

$$Q_1^n(\omega,x_1,x_2)=\frac{1}{2\pi}\int_{-\infty}^{\infty}G_1^n(\omega,\xi_1,x_2)\sum_m\exp[-i\xi_1(x_1-ml)-im\alpha]d\xi_1$$

(the prime on the summation sign denotes the absence of the $m=0$ term).

Thus we have

$$\Gamma_1^n(\omega,\mathbf{x})=\sum_{l=1}^{4}\Gamma_{1,l}^n(\mathbf{x}),\quad \Gamma_{11}^n(x)=G_1^n(0,\mathbf{x}),\quad \Gamma_{12}^n(x)=S^n(\mathbf{x}) \tag{2.12}$$

$$\Gamma_{13}^n(\mathbf{x})=-Q_1^n(0,\mathbf{x}),\quad \Gamma_{14}^n(\mathbf{x})=Q_1^n(\omega,\mathbf{x}),\quad \mathbf{x}=(x_1,x_2)$$

Using the formula

$$\sum_m\exp(iml\xi)=\frac{2\pi}{l}\sum_k\delta\left(\xi-k\frac{2\pi}{l}\right)$$

we obtain from (2.11) that

$$Q^n(\omega,x_1,x_2)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_m[G_1^n(\omega,\xi_1,x_2)-G_1^n(0,\xi_1,x_2)]\times$$

$$\times\exp[-i\xi_1(x_1-ml)-im\alpha]d\xi_1=\frac{1}{l}\sum_k[G_1^n(\omega,\xi_{1,k},x_2)- \tag{2.13}$$

$$-G_1^n(\omega,\xi_{1,k},x_2)]\exp(-i\xi_{1,k}x_1),\quad \xi_{1,k}=(2k\pi-\alpha)/l$$

It follows from (2.12) that for values of ω which make γ_{jk} vanish (i.e. γ_j when $\xi_1=\xi_{1,k}$), $\eta_0(\omega,\ \xi_{1,k},\ x_2)$ and consequently also $\Gamma_1^n(\omega,\ x_1,\ x_2)$ take infinitely large values. We say that these frequencies are resonant, and denote them by ω_{jk}.

Equating (2.11) to zero, we obtain

$$\omega_{jk}=|\xi_{1,k}|[a_{22}/2+(-1)^j(a_{22}^2/4-a_{21})^{1/2}]^{1/2} \tag{2.14}$$

Note that at non-resonant values of ω, $\Gamma_1^n(\omega,\ x_1,\ x_2)$ has the same singularities as $G_1^n(0,\ x_1,\ x_2)$ because $Q^n(\omega,\ x_1,\ x_2)$ and $S^n(x_1,\ x_2)$ are continuous.

3. Consider an orthotropic homogeneous elastic plane with a periodic row of circular apertures of radius R whose centres lie along the x_1 axis separated from one another by distances l. We will solve the stationary quasi-periodic problem in which the load on the apertures is transformed by an irreducible representation of the translation group with parameter α $(|\alpha|\leqslant\pi)$

$$\mathbf{p}_{k+1}(\theta) = \exp(-im\alpha)\mathbf{p}_k(\theta)$$

Here k is the number of the aperture ($k = 0, \pm 1, \pm 2, \ldots$) and θ is the angular coordinate of a point of the contour.

We will look for a solution of the problem in the form of a simple layer potential

$$\mathbf{u}(\mathbf{x}) = \int_\Pi \Gamma(\omega, \mathbf{x} - \mathbf{y})\mathbf{q}(\mathbf{y})ds_\mathbf{y} \tag{3.1}$$

($\mathbf{q}(\mathbf{y})$ is a two-dimensional vector and Π is the contour of the fundamental aperture whose centre lies at the origin of coordinates).

We will act on (3.1) with the operator $\mathbf{T}^\mathbf{n}$

$$\mathbf{T}^\mathbf{n}\mathbf{u}(\mathbf{x}) = \int_\Pi \Gamma_1^\mathbf{n}(\omega, \mathbf{x} - \mathbf{y})\mathbf{q}(\mathbf{y})ds_\mathbf{y} \tag{3.2}$$

Passing to the limit as $\mathbf{x} \to \mathbf{x}_0 \in \Pi$, we obtain

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \int_\Pi \Gamma_1^\mathbf{n}(\omega, \mathbf{x} - \mathbf{y})\mathbf{q}(\mathbf{y})as_\mathbf{y} = \mathbf{p}_0(\mathbf{x}_0) \tag{3.3}$$

Here $\mathbf{n}$ is the normal to the circle Π at the point $\mathbf{x}_0$ and $\mathbf{p}_0(\mathbf{x}_0)$ is the specified load at the contour.

The limit of the integral in (3.3) is not equal to the integral along the circle at $\mathbf{x} = \mathbf{x}_0$ because of the singularity of the kernel $\Gamma_1^\mathbf{n}(\omega, \mathbf{x})$ at $\mathbf{x} = 0$. As has already been noted, this singularity is the same as for $\mathbf{G}_1^\mathbf{n}(0, \mathbf{x})$, hence we obtain the additional term in the limit of the integral over Π from the kernel $\mathbf{G}_1^\mathbf{n}(0, \mathbf{x})$.

We introduce the complex variables $z_j = x + id_jx_2$, $\tau_j = y_1 + id_jy_2$.

Then

$$\frac{x_1 - y_1}{d_j^2(x_2 - y_2)^2 + (x_1 - y_1)^2} = \frac{1}{2}\left(\frac{1}{z_j - \tau_j} + \frac{1}{z_j + \tau_j}\right) \tag{3.4}$$

Let θ be the angle giving the position of the point $\mathbf{y}$ on the circle Π. Then $y_1 = R\cos\theta$, $y_2 = R\sin\theta$, $\tau_j = R(\cos\theta + id_j\sin\theta)$.

From this we have

$$d\tau_j = \tau_j'(\theta)d\theta = (-y_2 + id_jy_1)d\theta \tag{3.5}$$

Because $ds_y = Rd\theta$, using (3.5) we obtain

$$ds_y = r(\tau_j)d\tau_j, \quad r(\tau_j) = R/(y_2 + id_jy_1)$$

It can be shown that

$$\int_\Pi \frac{(x_1 - y_1)\mathbf{q}(\mathbf{y})ds_\mathbf{y}}{d_j^2(x_2 - y_2)^2 + (x_1 - y_1)^2} = \frac{1}{2}\left[\int_{\Pi_j} \frac{\mathbf{q}(\tau_j)r(\tau_j)}{z_j - \tau_j}d\tau_j + \int_{\overline{\Pi}_j} \frac{\mathbf{q}(\mu_j)r(\mu_j)}{\bar{z}_j - \mu_j}d\mu_j\right] \tag{3.6}$$

where $\mathbf{q}(\tau_j) = \mathbf{q}(\mu_j) = \mathbf{q}(\mathbf{y})$ and Π_j is the ellipse that is the image of the circle Π under the transformation $\tau_j = \tau_j(z)$, $z = y_1 + iy_2 \in \Pi$.

Similarly, $\overline{\Pi}_j$ is the image of Π under the mapping $\mu_j = \bar{\tau}_j(z)$, with the integration around $\overline{\Pi}_j$ being taken in the clockwise direction.

Assuming that the function $q = q(\mathbf{y})$ satisfies the Lipschitz–Hölder condition, from the Sokhotskii–Plemel' formulae we find that

$$\lim_{z\to z_{0j}} \int_{\Pi_j} \frac{\mathbf{q}(\tau_j)r(\tau_j)}{z_j-\tau_j}d\tau_j = \int_{\Pi_j} \frac{\mathbf{q}(\tau_j)r(\tau_j)}{z_{0j}-\tau_j}d\tau_j + \pi i\mathbf{q}(z_{0j})r(z_{0j}) \tag{3.7}$$

($z_{0j} \in \Pi_j$, $z_j \to z_{0j}$ from outside the edge Π_j).

In a similar manner one can obtain an expression for the limit of the second integral on the right-hand side of (3.6).

Using representations (2.10) and (2.11), and also the expression for the limits in formula (3.7), we have from (3.3) that

$$\mathbf{A}^{\mathbf{n}}(\mathbf{x})\mathbf{q}(\mathbf{x}) + \int_{\Pi} \Gamma_1^{\mathbf{n}}(\omega, \mathbf{x}-\mathbf{y})\mathbf{q}(\mathbf{y})ds_{\mathbf{y}} = \mathbf{p}_0(\mathbf{x}), \quad \mathbf{x}\in\Pi \tag{3.8}$$

$$\mathbf{A}^{\mathbf{n}}(\mathbf{x}) = d_0\pi R\sum_{j=1}^{2}(-1)^{j-1}[\mathbf{N}_{j1}^{\mathbf{n}}d_jx_1 - \mathbf{N}_{j2}^{\mathbf{n}}x_2/d_j]/(x_2^2+d_j^2x_1^2)$$

$$d_0 = [2a_0\pi(d_2^2-d_1^2)]^{-1}$$

This is the integral equation of the problem.

Below, if the vector $\mathbf{n}$ is directed along the normal to the edge Π, the superscript $\mathbf{n}$ will be omitted.

4. Replacing $\mathbf{x}$ by $\mathbf{x}-\mathbf{y}$ in (2.8) and substituting $y_1 = R\cos\theta$, $y_2 = R\sin\theta$, $x_1 = R\cos\theta_0$, $x_2 = R\sin\theta_0$, we have after some reduction

$$\mathbf{G}_1^{\mathbf{n}}(0,\mathbf{x}-\mathbf{y}) = \frac{d_0}{2R}\sum_{j=1}^{2}(-1)^{j-1}d_j\left[\mathbf{N}_{j1}^{n}\left(\cos\theta_0 + \operatorname{ctg}\frac{\theta-\theta_0}{2}\sin\theta_0\right)+\right.$$
$$\left.+\mathbf{N}_{j2}^{n}\left(\operatorname{ctg}\frac{\theta-\theta_0}{2}\cos\theta_0 - \sin\theta_0\right)\right]\left(d_j^2\cos^2\frac{\theta_0+\theta}{2}+\sin^2\frac{\theta_0+\theta}{2}\right)^{-1} \tag{4.1}$$

From (2.10), (2.11) and (4.11) it follows that Eq. (3.8) is singular. We shall seek its solution in the form

$$\mathbf{q}(\theta) = \sum_m \mathbf{q}_m\exp(im\theta) \tag{4.2}$$

Henceforth the limit of $\mathbf{q}(\mathbf{y})$ as $\mathbf{y}\to\Pi$ is denoted by $q(\theta)$ where θ is the polar angle of the point $\mathbf{y}$. Similarly $p_0(\mathbf{x}) = p_0(\theta_0)$ (θ_0 being the polar angle of the point $\mathbf{x}$), $\Gamma_1(\omega, \mathbf{x}-\mathbf{y}) = \Gamma_1(\omega, \theta_0, \theta)$, etc.

Substituting (4.2) into (3.9), we obtain a modified equation; multiplying both sides of it by $(2\pi)^{-1}\exp(-in\theta_0)$ and integrating over θ_0 from $-\pi$ to π we arrive at an infinite system of linear algebraic equations with respect to $\mathbf{q}_m$

$$\sum_m \mathbf{B}_{nm}\mathbf{q}_m = \mathbf{p}_n \tag{4.3}$$

Here

$$\mathbf{B}_{nm} = \mathbf{A}_{n-m} + \frac{R}{2\pi}\sum_{l=1}^{4}\mathbf{B}_{lnm}$$

$$\mathbf{A}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathbf{A}(\theta_0)\exp(-in\theta_0)d\theta_0, \quad \mathbf{p}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathbf{p}_0(\theta_0)\exp(-in\theta_0)d\theta_0$$

$$\mathbf{B}_{lnm} = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Gamma_{1,l}(\theta_0,\theta)\exp(im\theta - in\theta_0)d\theta d\theta_0 \tag{4.4}$$

Without dwelling on their derivation we give formulae for computing the coefficients of the linear algebraic system

$$\mathbf{B}_{lnm}=\frac{1}{2}\sum_{j=1}^{2}(-1)^{j-1}\sum_{k=1}^{2}[w_{j,2l-2+k,n-1,m}(\mathbf{N}_{jk2}-i\mathbf{N}_{jk1})+$$
$$+w_{j,2l-2+k,n+1,m}(\mathbf{N}_{jk2}+i\mathbf{N}_{jk1})] \tag{4.5}$$

Here $\mathbf{N}_{jkm}$ (j, k, $m=1$, 2) are two-dimensional matrices which are coefficients of the following representation of the matrix $\mathbf{N}_{jk}^{\mathbf{n}}$

$$\mathbf{N}_{jk}^{\mathbf{n}}=\mathbf{N}_{jk1}\sin\theta_0+\mathbf{N}_{jk2}\cos\theta_0,\quad \mathbf{n}=(\cos\theta_0,\sin\theta_0)$$
$$w_{j,1,n,m}=d_0\pi^2R^{-1}[s_{j,2m}(\delta_{n+1,m}+\delta_{n-1,m})+$$
$$+s_{j,n-m-1}\operatorname{sign}(m+n-1)-s_{j,n-m+1}\operatorname{sign}(m+n+1)]$$
$$w_{j,2,n,m}=id_0\pi^2R^{-1}[s_{j,2m}(\delta_{n-1,m}-\delta_{n+1,m})+$$
$$+s_{j,n-m-1}\operatorname{sign}(m+n-1)+s_{j,n-m+1}\operatorname{sign}(m+n+1)]$$
$$w_{j,3,n,m}=4d_0\pi^2R^{-1}\sum_{p}e^{-ip\alpha}(t_{j,p,m+1}+t_{j,p,m-1}-plR^{-1}t_{jpm})\delta_{nm} \tag{4.6}$$
$$w_{j,4,n,m}=i4d_0\pi^2R^{-1}\sum_{p}e^{-ip\alpha}(t_{j,p,m-1}-t_{j,p,m+1})\delta_{nm}$$
$$w_{j,5,n,m}=-4\pi^2a_0l^{-1}\sum_{q,p}\operatorname{sign}(q)f_{jqp}\exp[i(m-n)\alpha_{pq}]J_m(a_{pq})J_n(a_{pq})$$
$$w_{j,6,n,m}=-4\pi^2a_0l^{-1}\sum_{q,p}g_{jqp}\exp[i(m-n)\alpha_{pq}]J_m(a_{pq})J_n(a_{pq})$$

($J_m(x)$ is the Bessel function of the first kind and δ_{nm} is the Kronecker delta).

In these formulae s_{jm}, t_{jkm}, f_{jkm} and g_{jkm} are the coefficients of the expansions

$$(d_j^2\cos^2\theta+\sin^2\theta)^{-1}=\sum_{m}s_{jm}e^{im\theta},\quad [d_j^2\sin^2\theta+(kl-\cos\theta)^2]^{-1}=\sum_{m}t_{jkm}e^{im\theta}$$
$$\exp(-d_j|\xi_{1,k}x_2|)=\sum_{m}f_{jkm}\exp(im\pi x_2/2R)$$
$$\exp(-d_j|\xi_{1,k}x_2|)\operatorname{sign}x_2=\sum_{m}g_{jkm}\exp(im\pi x_2/2R)$$

The quantities a_{jk} and α_{jk} are defined by the equations

$$a_{jk}=(j^2\pi^2/4+R^2\xi_{1,k}^2)^{1/2},\quad \operatorname{tg}\alpha_{jk}=2R\xi_{1,k}/j\pi$$

It can be shown that

$$s_{jm}=O(z_1^{|m|}),\quad t_{jkm}=O(z_2^{|m|}),\quad f_{jkm}=O(m^{-2}),\quad g_{jkm}=O(|m|^{-1}) \tag{4.7}$$

where the z_k are complex numbers such that $|z_k|<1$ ($k=1$, 2).

We note that the relations

$$\int_{-\pi}^{\pi}\operatorname{ctg}\left(\frac{\theta-\theta_0}{2}\right)\exp(im\theta)d\theta=2\pi i\exp(im\theta_0)\operatorname{sig}nm$$

were used in deriving (4.5) and (4.6).

Similar relations also hold for $\mathbf{A}_n$

$$\mathbf{A}_n=\frac{1}{2}\sum_{j=1}^{2}(-1)^{j-1}\sum_{k=1}^{2}[w_{j,6+k,n-1}(\mathbf{N}_{j,k,2}+\mathbf{N}_{j,k,1})+w_{j,6+k,n+1}(\mathbf{N}_{j,k,2}-\mathbf{N}_{j,k,1})]$$

$$w_{j,7,n}=\frac{d_0d_1}{2}(s_{j,n-1}+s_{j,n+1}),\quad w_{j,8,n}=\frac{d_0}{2iRd_j}(s_{j,n-1}-s_{j,n+1})$$

5. We will show that for non-resonant values of ω the system reduces to a quasi-regular one. From (4.4) and (4.5) we have

$$\begin{aligned}\mathbf{B}_{1,n,m}&=\frac{d_0\pi^2}{2R}\sum_{j=1}^{2}(-1)^{j-1}d_j[s_{j,2,m}(\delta_{n-2,-m}\mathbf{E}_{j1}+\\&+\delta_{n+2,-m}\mathbf{E}_{j2}+2\delta_{n,-m}\mathbf{E}_{j3})+s_{j,n-m-2}\operatorname{sign}(m+n-2)\mathbf{E}_{j1}-\\&-s_{j,n-m+2}\operatorname{sign}(m+n+2)\mathbf{E}_{j2}+2s_{j,n-m}\operatorname{sign}(m+n)\mathbf{E}_{j0}\end{aligned}\tag{5.1}$$

Here

$$\mathbf{E}_{j0}=i(\mathbf{N}_{j11}-\mathbf{N}_{j22}),\quad \mathbf{E}_{j1}=-i\mathbf{N}_{j11}+\mathbf{N}_{j12}+\mathbf{N}_{j21}-i\mathbf{N}_{j22}$$
$$\mathbf{E}_{j2}=i\mathbf{N}_{j11}+\mathbf{N}_{j12}+\mathbf{N}_{j21}+i\mathbf{N}_{j22},\quad \mathbf{E}_{j3}=\mathbf{N}_{j12}-\mathbf{N}_{j21}$$

It can be shown that for $|n|\geqslant 2$

$$\mathbf{B}_{1,n,n}=\mathbf{B}_0\operatorname{sign}n,\quad \mathbf{B}_0=\frac{d_0\pi^2}{2R}\sum_{j=1}^{2}(-1)^{j-1}[s_{j2}(\mathbf{E}_{j1}-\mathbf{E}_{j2})+2s_{j0}\mathbf{E}_{j0}]\tag{5.2}$$

Similarly

$$\mathbf{A}_n=\frac{d_0\pi}{4}\sum_{j=1}^{2}(-1)^{j-1}d_j(s_{j,n-2}\mathbf{E}_{j5}+s_{j,n+2}\mathbf{E}_{j6}+2s_{jn}\mathbf{E}_{j4})$$
$$\mathbf{E}_{j4}=d_j\mathbf{N}_{j12}+d_j^{-1}\mathbf{N}_{j21},\quad \mathbf{E}_{j5}=d_j(-i\mathbf{N}_{j11}+\mathbf{N}_{j12})-d_j^{-1}(\mathbf{N}_{j21}+i\mathbf{N}_{j22})$$
$$\mathbf{E}_{j6}=d_j(i\mathbf{N}_{j11}+\mathbf{N}_{j22})-d_j^{-1}(\mathbf{N}_{j21}-i\mathbf{N}_{j22})$$

From this

$$\mathbf{A}_0=\frac{d_0\pi}{4}\sum_{j=1}^{2}(-1)^{j-1}d_j[s_{j2}(\mathbf{E}_{j5}+\mathbf{E}_{j6})+2s_{j0}\mathbf{E}_{j4}]$$

We introduce the notation $\mathbf{D}_n=\mathbf{A}_0+R\mathbf{B}_0\operatorname{sign}n/(2\pi)$ and make the substitution $\mathbf{q}_m=\mathbf{D}_m^{-1}\mathbf{q}_m^D$ in system (4.3). As a result we obtain the system

$$\sum_m\mathbf{B}_{nm}^D\mathbf{q}_m^D=\mathbf{p}_n,\quad \mathbf{B}_{nm}^D=\mathbf{B}_{nm}\mathbf{D}_m^{-1}\tag{5.3}$$

We note that

$$\mathbf{B}_{nm}^D=\sum_{l=1}^{4}\mathbf{B}_{lnm}^D,\mathbf{B}_{1,n,m}^D=\left(\mathbf{A}_{n-m}+\frac{R}{2\pi}\mathbf{B}_{1,n,m}\right)\mathbf{D}_m^{-1},\quad \mathbf{B}_{lnm}^{-1}=\mathbf{B}_{lnm}\mathbf{D}_m^{-1}\ (l>1)$$

Obviously, for $|n|\geqslant 2$ the main diagonal of the matrix $\mathbf{B}_{1,n,m}^D$ consists of ones.

We estimate the sum

$$\sum_{l=1}^{2}\sum_{m\neq n}|\mathbf{B}^D_{1,n,m,k,l}|$$

(where the $\mathbf{B}^D_{1,n,m,k,l}$ are elements of the matrix $\mathbf{B}^D_{1,n,m}$).

From (5.1) it follows that

$$\sum_{l=1}^{2}\sum_{m\neq n}\left|\mathbf{B}^D_{1,n,m,k,l}\right| \leq \beta_{nk} \tag{5.4}$$

$$\beta_{nk}=\frac{d_0\pi}{4}\sum_{j=1}^{2}d_j\sum_{l=1}^{2}[\left|\mathbf{E}^D_{1,j,k,l}\right|\left|s_{j,2n-4}\right|+\left|\mathbf{E}^D_{j,2,n,l}\right|\left|s_{j,2n+4}\right|+$$
$$+2\left|\mathbf{E}^D_{j,3,k,l}\right|\left|s_{jn}\right|+(\left|\mathbf{E}^D_{j,1,k,l}\right|+\left|\mathbf{E}^D_{j,5,k,l}\right|)\sum_m{}'\left|s_{j,m-2}\right|+$$
$$+(\left|\mathbf{E}^D_{j,2,k,l}\right|+\left|\mathbf{E}^D_{j,6,k,l}\right|)\sum_m{}'\left|s_{j,m+2}\right|+(\left|\mathbf{E}^D_{j,0,k,l}\right|+\left|\mathbf{E}^D_{j,4,k,l}\right|)\sum_m{}'\left|s_{jm}\right|$$

Here the prime on the summation sign denotes the absence of the $m=n$ term, and $\mathbf{E}^D_{jmkl}$ are elements of the matrices $\mathbf{E}^D_{jm}=\mathbf{E}_{jm}D^{-1}_m$.

The convergence of the series in (5.4) follows from the asymptotic estimates (4.7).

The condition $\beta_{nk}<1$ together with the condition for $\mathbf{D}^{-1}_m$ to exist are restrictions imposed on the elastic constants of the material. For orthotropic materials whose properties were given in [6, 7] these conditions are, however, satisfied.

It follows from (4.7) that the coefficients $\mathbf{B}^D_{2,n,m}$ have no effect on the quasi-regularity of system (5.3).

One can show that the function $\mathbf{Q}(\omega,\theta_0,\theta)$ is continuous on the square $|\theta_0|$, $|\theta|\leq\pi$, where it has partial derivatives satisfying a Lipschitz condition of some order μ $(0<\mu<1)$. Integrating by parts over θ in (4.4) and noting that the non-integral terms vanish, we obtain

$$\mathbf{B}_{3,n,m}+\mathbf{B}_{4,n,m}=\frac{i}{m}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\partial}{\partial\theta}[\mathbf{Q}(\omega,\theta_0,\theta)-\mathbf{Q}(0,\theta_0,\theta)]e^{im\theta}d\theta e^{-in\theta_0}d\theta_0 \tag{5.5}$$

Because $\partial/\partial\theta[\mathbf{Q}(\omega,\theta_0,\theta)-\mathbf{Q}(0,\theta_0,\theta)]$ satisfies the Lipschitz condition, the inner integral tends to zero like $|m|^{-\mu}$ as $|m|\to\infty$. The inner integral is, moreover, a continuous function of θ_0. Hence the right-hand side of (5.5) also tends to zero as $|n|\to\infty$. It then follows from (5.5) that

$$S^D_{nk}=\sum_m\sum_{l=1}^{2}\left|\mathbf{B}^D_{3,n,m,k,l}+\mathbf{B}^D_{4,n,m,k,l}\right|$$

exists and $S^D_{nk}\to 0$ as $|n|\to\infty$.

Summarizing all the above concerning the coefficients of system (5.3), we arrive at the conclusion that this system is quasi-regular, which means that both it and system (4.3) can be solved by reduction.

6. We will investigate the behaviour of the solution of (3.9) in the neighbourhood of the resonant frequency (2.14). As an example we will assume that $\omega\to\omega_{1,k}$ (where k is fixed). Then $\gamma_{1,k}\to 0$, while the term in $\mathbf{G}^*_1(\omega,\xi_{1,k},x_2)$ tends to infinity. It can be shown that this term has the form given below and the following asymptotic behaviour

$$\mathbf{U}_0(\omega,\xi_{1,k})\frac{\exp(-\gamma_{1,k}|x_2|)}{2a_0l(\gamma^2_{2,k}-\gamma^2_{1,k})\gamma_{1,k}}\sim\mathbf{U}_0(\omega_{1,k},\xi_{1,k})[2a_0l\gamma^2_{2,k}\gamma_{1,k}]^{-1}$$

as $\omega\to\omega_{1,k}$, $\gamma_{1,k}\to 0$.

Then Green's stress tensor for the quasi-periodic problem can be written as follows:

$$\Gamma_1^n(\omega,x_1,x_2)=\mathbf{G}_1^n(0,x_1,x_2)+\mathbf{U}_0(\omega_{1,k},\xi_{1,k})\frac{\exp(-i\xi_{1,k}x_1)}{2a_0l\gamma_{2,k}^2}\gamma_{1,k}^{-1}+\mathbf{H}_k^n(\omega,x_1,x_2) \tag{6.1}$$

where $\mathbf{H}_k^n(\omega, x_1, x_2)$ is a matrix that is continuous as a function of x_1 and x_2 and remains bounded as $\omega\to\omega_{1,k}$.

With the help of (6.1) the integral equation (3.9) can be written as

$$(\varepsilon^{-1}\mathbf{V}_1+\mathbf{V}_2)\mathbf{q}=\mathbf{v} \tag{6.2}$$

$$\varepsilon=\gamma_{1,k},\quad \mathbf{V}_1\mathbf{q}=(2\pi)^{-1}\int_{-\pi}^{\pi}\exp[-i\xi_{1,k}\mathrm{R}(\cos\theta_0-\cos\theta)]\mathbf{q}(\theta)d\theta$$

$$\mathbf{V}_2\mathbf{q}=\frac{a_0l\gamma_{2,k}^2}{\pi}\mathbf{U}_0^{-1}(\omega_{1,k}\xi_{1,k})\left[\mathbf{A}(x)\mathbf{q}(x)+\int_\Pi \mathbf{G}_1(0,\mathbf{x}-\mathbf{y})\mathbf{q}(\mathbf{y})ds_y+\right.$$
$$\left.+\int_\Pi H_k(\omega,\mathbf{x}-\mathbf{y})\mathbf{q}(\mathbf{y})ds_y\right],\quad \mathbf{v}=\frac{a_0l\gamma_{2,k}^2}{\pi}\mathbf{U}_0^{-1}(\omega_{1,k},\xi_{1,k})\mathbf{p}_0(x) \tag{6.3}$$

Note that $\mathbf{V}_1$ is a finite-dimensional operator transferring the space $L_2^{(2)}(-\pi,\pi)$ into the two-dimensional subspace spanned by the vectors $\mathbf{e}_j\exp(-i\xi_{1,k}R\cos\theta_0)$ $(j=1, 2)$. $L_2^{(2)}(-\pi,\pi)$ is the Hilbert space of square-integrable two-dimensional vectors with scalar product

$$(\mathbf{p},\mathbf{q})=\int_{-\pi}^{\pi}\sum_{j=1}^{2}p_j(\theta)\bar{q}_j(\theta)d\theta$$

The operator $\mathbf{V}_2$ depends continuously on ε, hence for small ε we put $\mathbf{V}_2(\varepsilon)=\mathbf{V}_2(0)$. From (6.2) we obtain

$$(\mathbf{V}_1+\varepsilon\mathbf{V}_2)\mathbf{q}=\varepsilon\mathbf{v} \tag{6.4}$$

We note that $L_2^{(2)}(-\pi,\pi)$ can be represented in the form of an orthogonal sum of subspaces

$$L_2^{(2)}(-\pi,\pi)=R+N \tag{6.5}$$

Here R is the domain of values of the self-conjugate operator V_1, while N is its null-space. It follows from (6.5) that one can construct an orthonormal basis in $L_2^{(2)}(-\pi,\pi)$ as follows:

$$\mathbf{g}_{0j}\in R,\quad \mathbf{g}_{mj}(m\neq0)\in N\quad (j=1,2)$$

For example, one can put $\mathbf{g}_{mj}=\mathbf{e}_j\exp(-i\xi_{1,k}R\cos\theta-im\theta)$.

We will seek a solution of Eq. (6.4) in the form of a series

$$\mathbf{q}(\theta)=\sum_m\sum_{j=1}^{2}p_{mj}\mathbf{g}_{mj}(\theta) \tag{6.6}$$

Substituting (6.6) into (6.4) and scalar-multiplying by $\mathbf{g}_{nl}(\theta_0)$, we obtain

$$p_{0l}+\varepsilon\sum_m\sum_{j=1}^{2}p_{mj}(\mathbf{V}_2\mathbf{g}_{mj},\mathbf{g}_{0l})=\varepsilon(\mathbf{v},\mathbf{g}_{0l})\quad (l=1,2)$$
$$\sum_m\sum_{j=1}^{2}p_{mj}(\mathbf{V}_2\mathbf{g}_{mj},\mathbf{g}_{nl})=(\mathbf{v},\mathbf{g}_{nl})\quad (n\neq0,\ l=1,2) \tag{6.7}$$

We eliminate p_{0l} $(l=1, 2)$ from (6.7) and arrive at the following system of linear equations for p_{mj} $(j=1, 2;\ m=\pm1, \pm2, \ldots)$

$$\sum_{m\neq 0}\sum_{j=1}^{2} V_{mjnl}p_{mj} = \upsilon_{nl} \quad (n=\pm1,\pm2,\ldots; l=1,2)$$

Here

$$V_{mjnl} = (\mathbf{V}_2\mathbf{g}_{mj}, \mathbf{g}_{nl}) + \varepsilon\sum_{p=1}^{2}(\mathbf{V}_2\mathbf{g}_{mj}, \mathbf{g}_{0p})\kappa_{pnl}$$

$$\kappa_{1,n,l} = \{-[1+\varepsilon(\mathbf{V}_2\mathbf{g}_{02},\mathbf{g}_{02})](\mathbf{V}_2\mathbf{g}_{01},\mathbf{g}_{nl}) + \varepsilon(\mathbf{V}_2\mathbf{g}_{01},\mathbf{g}_{02})(\mathbf{V}_2\mathbf{g}_{02},\mathbf{g}_{nl})\}/\kappa$$

$$\kappa_{2,n,l} = \{\varepsilon(\mathbf{V}_2\mathbf{g}_{02},\mathbf{g}_{01})(\mathbf{V}_2\mathbf{g}_{01},\mathbf{g}_{nl}) + [1-\varepsilon(\mathbf{V}_2\mathbf{g}_{01},\mathbf{g}_{01})](\mathbf{V}_2\mathbf{g}_{02},\mathbf{g}_{nl})\}/\kappa$$

$$\kappa = 1+\varepsilon[(\mathbf{V}_2\mathbf{g}_{01},\mathbf{g}_{01}) + (\mathbf{V}_2\mathbf{g}_{02},\mathbf{g}_{02})] + \varepsilon^2[(\mathbf{V}_2\mathbf{g}_{01},\mathbf{g}_{01})(\mathbf{V}_2\mathbf{g}_{02},\mathbf{g}_{02}) -$$

$$-(\mathbf{V}_2\mathbf{g}_{01},\mathbf{g}_{02})(\mathbf{V}_2\mathbf{g}_{02},\mathbf{g}_{01})], \quad \upsilon_{nl} = (\mathbf{v},\mathbf{g}_{nl}) + \varepsilon\sum_{p=1}^{2}(\mathbf{v},\mathbf{g}_{0p})\kappa_{pnl}$$

It follows from (6.3) that system (6.6) becomes quasi-regular for sufficiently small ε (including $\varepsilon=0$). The proof is performed as for (4.4). Thus the behaviour of the solution in the neighbourhood of a resonant point is in principle no different from its behaviour at other frequencies.

Example. A periodic problem $(\alpha=0)$. Suppose that the same normal load $p(\omega, t)=\exp(-i\omega t)$ acts at the contours of all the apertures. For a plastic that is 2:1 perpendicularly glass-reinforced [7] with $l=4$ we have the following values for the resonant frequencies: $\omega_{1,k}=3.025k$ and $\omega_{2,k}=1.111k$ $(k=1, 2, \ldots)$; while for $l=6$, $\omega_{1,k}=2.017k$ and $\omega_{2,k}=0.741k$. The quantities l and ω are dimensionless: $l=l'/R$, $\omega^2=4\rho(\omega' R)^2 c_{66}^{-1}$ (l', ω', R, ρ and c_{66} are dimensionless quantities). Figure 2 shows graphs of $|\sigma_\theta|_{\max}/(2c_{66})$ as a function of ω on the contour of the aperture for $l=4$ (the solid line) and for $l=6$ (the dashed line). There are finite maxima near the resonant frequencies. They decrease and their abscissae approach $\omega_{l,k}$ as ω increases. Figure 3 shows the evolution of the $|\sigma_\theta|$ diagram on the contour of the aperture when the resonant frequency $\omega_{1,1}$ is approached for $l=6$.

If a plane wave is incident on the periodic system, then one can verify that at points of two adjacent apertures the stresses corresponding to the reflected wave differ by a factor $\exp(i\kappa l\cos\phi)$ (where ϕ is the angle of incidence of the wave and κ is its wave number). Thus in this case we have a quasi-periodic problem with $\alpha=-\kappa/\cos\phi$.

If one considers a radiation problem with a load on the apertures that does not possess quasi-periodic properties, then its solution reduces to the solution of a set of quasi-periodic problems and their superposition.

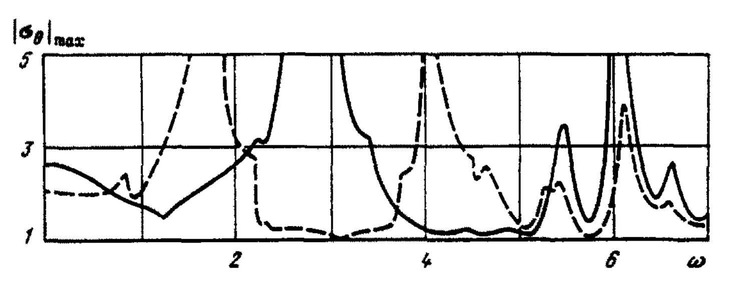

FIG. 2.

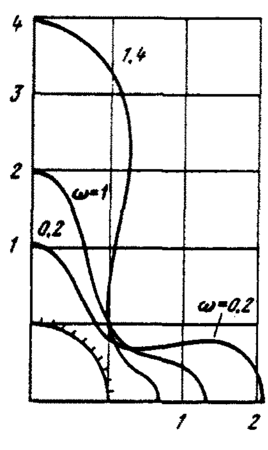

FIG. 3.

This work was performed under the auspices of the "Non-regularity" project financed by the Ukrainian Government Committee for Science and Technology.

REFERENCES

1. FOMIN V. M., Steady-state problems of the radiation and diffraction of elastic waves in a plane with a periodic series of apertures. *Izv. Akad. Nauk SSSR, MTT* **2**, 75–91, 1991.
2. GUZ' A. N. and GOLOVCHAN V. T., *Diffraction of Elastic Waves in Multiply-connected Bodies.* Nauk. Dumka, Kiev, 1972.
3. GUZ' A. N., KUBENKO V. D. and CHEREVKO M. A., *Diffraction of Elastic Waves.* Nauk. Dumka, Kiev, 1973.
4. CHEREVKO M. A., Diffraction of a longitudinal wave by a series of hollow circular inclusions. *Prikl. Mekh.* **18**, 2, 120–123, 1982.
5. KOSMODAMIANSKII A. S. and STOROZHEV V. I., *Dynamical Problems of the Theory of Elasticity for Anisotropic Media.* Nauk. Dumka, Kiev, 1985.
6. FEDOROV F. I., *Elastic Wave Theory of Crystals.* Nauka, Moscow, 1986.
7. LEKHNITSKII S. G., *Theory of Elasticity for an Anisotropic Body.* Nauka, Moscow, 1977.
8. BALAKIREV M. K. and GILINSKII I. A., *Waves in Piezoelectric Crystals.* Nauka, Novosibirsk, 1982.
9. NATROSHVILI D. G., The fundamental matrices of the equations of steady-state oscillations and pseudo-oscillations in the anisotropic theory of elasticity. *Soobshch. Akad. Nauk. Gruz. SSR* **96**, 1, 49–52, 1979.
10. KUPRADZE V. D., *Potential Methods in the Theory of Elasticity.* Fizmatgiz, Moscow, 1963.
11. TITCHMARSH E., *Theory of Functions* (in Russian). Nauka, Moscow, 1980.
12. FOMIN V. M., A plane steady-state dynamical contact problem for a periodic structure. *Izv. Akad. Nauk SSSR, MTT* 1, 84–94, 1987.

Translated by R.L.Z.